STRONG ALGEBRABILITY OF SETS OF SEQUENCES AND FUNCTIONS

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ABSTRACT. We introduce a notion of strong algebrability of subsets of linear algebras. Our main results are the following. The set of all sequences from c_0 which are not summable with any power is densely strongly $\mathfrak{c}$–algebrable. The set of all sequences in l^∞ whose sets of limit points are homeomorphic to the Cantor set is comeager and strongly $\mathfrak{c}$-algebrable. The set of all non-measurable functions from $\mathbb{R}^{\mathbb{R}}$ is strongly $2^{\mathfrak{c}}$–algebrable. These results complete several ones by other authors, within the modern context of lineability.

1. INTRODUCTION

Assume that B is a linear algebra, that is, a linear space being also an algebra. According to [4], a subset $E \subset B$ is called algebrable whenever $E \cup \{0\}$ contains an infinitely generated algebra. And if κ is a cardinal, E is said to be a κ–algebrable if $E \cup \{0\}$ contains an algebra generated by an algebraically independent set with cardinality κ. Hence algebrability equals ω-algebrability, where ω denotes the cardinality of the set $\mathbb{N}$ of positive integers. In addition to algebrability, there exist other related properties such as lineability, dense-lineability and spaceability, see [3], [8], [9] and [14].

In this note we will focus on algebrability. This notion was considered by many authors. For instance Aron, Pérez-García and Seoane-Sepúlveda proved in [4] that the set of non-convergent Fourier series is algebrable. Aron and Seoane-Sepúlveda established in [5] algebrability of everywhere

surjective functions on $\mathbb{C}$, which was strengthened in [2]. Other algebrability results can be found in [11] and [12]. Here we will discuss some strengthening of the notion of algebrability.

For a cardinal κ, we say that A is a κ-generated free algebra, if there exists a subset $X = \{x_\alpha : \alpha < \kappa\}$ of A such that any function f from X to some algebra A', can be uniquely extended to a homomorphism from A into A'. Then X is called a set of free generators of the algebra A. A subset $X = \{x_\alpha : \alpha < \kappa\}$ of a commutative algebra B generates a free sub-algebra A if and only if for each polynomial P and any $x_{\alpha_1}, x_{\alpha_2}, ..., x_{\alpha_n}$ we have $P(x_{\alpha_1}, x_{\alpha_2}, ..., x_{\alpha_n}) = 0$ if and only if $P = 0$.

This leads us to the following definition. We say that a subset E of a commutative linear algebra B is *strongly κ–algebrable*, if there exists a κ-generated free algebra A contained in $E \cup \{0\}$. If additionally, B is equipped with some topological structure and A is dense in B, then we say that E is *densely strongly κ–algebrable*.

Note that $X = \{x_\alpha : \alpha < \kappa\} \subset E$ is a set of free generators of a free algebra $A \subset E$ if and only if the set $\tilde{X}$ of elements of the form $x_{\alpha_1}^{k_1} x_{\alpha_2}^{k_2} \cdots x_{\alpha_n}^{k_n}$ is linearly independent and all linear combinations of elements from $\tilde{X}$ are in $E \cup \{0\}$. (In particular, it follows that sets considered in [7] were strongly algebrable.) It easy to see that free algebras have no divisors of zero.

In general, there are subsets of linear algebras which are algebrable but not strongly algebrable. Let c_{00} be a subset of c_0 consisting of all sequences with real terms equal to zero from some place. Having members $x_1, ..., x_n$ of some linear algebra and a polynomial P, an element $P(x_1, ..., x_n)$ will be called an algebraic combination of $x_1, ..., x_n$.

Proposition 1. *The set c_{00} is algebrable in c_0 but is not strongly 1–algebrable.*

Proof. Note that c_{00} is a linear sub-algebra of c_0 which can be generated by a linearly independent set of cardinality ω, namely, by $\{e_1, e_2, ...\}$ where e_i is a sequence consisting of 1 on the i-th coordinate and zeros on the

remaining coordinates. Note also that c_{00} cannot be generated by finitely many elements, since any linear (or algebraic) combination of finitely many elements from c_{00} eventually equals zero.

Suppose that c_{00} is strongly 1–algebrable, i.e. there is a sequence $a \in c_{00}$ which is algebraically independent. In particular, it means that the collection $\{a, a^2, a^3, ...\}$ is linearly independent. Since $a \in c_{00}$, there is $n \in \mathbb{N}$ such that $a(k) = 0$ for $k \geq n$. Moreover $a^m(k) = 0$ for all $k \geq n$ and $m \in \mathbb{N}$. Note that the linear space generated by $a, a^2, a^3, ...$ is isomorphic to a linear subspace of $\mathbb{R}^n$ and restriction vectors $a_{\restriction n}, a^2_{\restriction n}, a^3_{\restriction n}, ...$ are linearly independent in $\mathbb{R}^n$. A contradiction. $\square$

The aim of this note is to prove strong $\mathfrak{c}$–algebrability of some sets of sequences and functions, where $\mathfrak{c}$ denotes cardinality of the continuum. These are: the set of all sequences tending to zero which are not summable with any power, the set of non-measurable functions, and the set of bounded real sequences whose set of accumulation points is of measure zero and can be decomposed into a finite set and a set homeomorphic to the Cantor set. This last set happens to be co-meager.

2. ALGEBRABILITY IN c_0 AND l^∞

To prove the following Theorem 2 we will use the idea from the paper [7]. Our theorem is related to [10, Theorem 1.3], from which the spaceability of $c_0 \setminus \bigcup \{l^p : p \geq 1\}$ is extracted as a consequence. Recall that a subset E of a topological vector space X is called spaceable if $E \cup \{0\}$ contains a closed infinite dimensional subspace of X.

Theorem 2. *The set $c_0 \setminus \bigcup \{l^p : p \geq 1\}$ is densely strongly $\mathfrak{c}$–algebrable in c_0.*

Proof. Let $\{r_\alpha : \alpha < \mathfrak{c}\}$ be a linearly independent (with respect to the field $\mathbb{Q}$) subset of positive reals containing 1. Let A be a linear algebra generated

by the set

$$S = \left\{ \left(\frac{1}{\ln^{r_\alpha}(n+1)} \right)_{n=1}^{\infty} : \alpha < \mathfrak{c} \right\}.$$

We will show that any non-trivial algebraic combination of elements from S is either a null sequence or it is not summable with any power. To prove this, we need to show that, for every matrix $(k_{il} : i \le m, l \le j)$ of non-negative integers with non-zero and distinct rows, and any $\beta_1, ..., \beta_m \in \mathbb{R}$ which do not vanish simultaneously, the series

(1)
$$\sum_{n=1}^{\infty} \left(\beta_1 \frac{1}{\ln^{k_{11}r_{\alpha_1} + ... + k_{1j}r_{\alpha_j}}(n+1)} + ... + \beta_m \frac{1}{\ln^{k_{m1}r_{\alpha_1} + ... + k_{mj}r_{\alpha_j}}(n+1)} \right)$$

is not summable with any power. Note that the sequence

(2)
$$\left(\frac{1}{\ln^{k_{11}r_{\alpha_1} + ... + k_{1j}r_{\alpha_j}}(n+1)} \right)_{n \in \mathbb{N}}$$

tends to zero, i.e. it is in c_0. We need to prove that the series (1) is divergent. Since the set $\{r_\alpha : \alpha < \mathfrak{c}\}$ is linearly independent, the numbers $k_{11}r_{\alpha_1} + ... + k_{1j}r_{\alpha_j}, ..., k_{m1}r_{\alpha_1} + ... + k_{mj}r_{\alpha_j}$ are distinct. To simplify the notation, put $k_i := k_{i1}r_{\alpha_1} + ... + k_{ij}r_{\alpha_j}$ for every $i = 1, ..., m$. We may assume that $k_1 < k_2 < ... < k_m$ and $\beta_1 \neq 0$. Since $\frac{\ln^{k_i}(n+1)}{\ln^{k_1}(n+1)} \to \infty$ $(n \to \infty)$ for all $i = 2, ..., m$, we obtain that there is N such that

(3)
$$\left| \beta_1 \frac{1}{\ln^{k_1}(n+1)} + \beta_2 \frac{1}{\ln^{k_2}(n+1)} + ... + \beta_m \frac{1}{\ln^{k_m}(n+1)} \right| \ge \frac{|\beta_1|}{2\ln^{k_1}(n+1)}$$

for all $n \ge N$. Since $\left(\frac{1}{\ln^q(n+1)} \right)_{n=1}^{\infty} \notin l^p$ for any $p \ge 1$ and $q > 0$, the result follows.

Now, we prove that A is dense in c_0. Consider the sub-algebra A_1 of space c (of all convergent sequences with the sup-norm), generated by two elements: $\left(\frac{1}{\ln(n+1)} \right)_{n=1}^{\infty}$ and $\mathbb{1} = (1,1,1,...)$. Let A_2 be a sub-algebra of A generated by $\left(\frac{1}{\ln(n+1)} \right)_{n=1}^{\infty}$. Note that $A_1 = \{x + (a,a,a,...) : x \in A_2, a \in \mathbb{R}\}$. Equip the set of positive integers $\mathbb{N}$ with the discrete topology and let $\alpha\mathbb{N}$ be a one-point compactification of $\mathbb{N}$. Note that c is isometrically isomorphic

to the space $C(\alpha\mathbb{N})$ of all continuous real functions on $\alpha\mathbb{N}$ with the sup-norm. Since $\left(\frac{1}{\ln(n+1)}\right)_{n=1}^{\infty}$ is strictly decreasing sequence of positive numbers tending to zero, then it separates the points of $\alpha\mathbb{N}$. Therefore by the Stone-Weierstrass theorem, the algebra A_1 is dense in c.

Take any $x \in c_0$ and $\varepsilon > 0$. There is $y \in A_1$ with $||x - y|| < \varepsilon/2$. Note that $|\lim_n y(n)| \le \varepsilon/2$. Put $\hat{y} = (y(1) - \lim_n y(n), y(2) - \lim_n y(n), y(3) - \lim_n y(n), ...) \in c_0$. Then $||x - \hat{y}|| < \varepsilon$. Since $\hat{y} \in A_2 \subset A$, we obtain that A_2, and therefore A, is dense in c_0. $\square$

In the sequel we will need the following observation. Let S be a set defined in the proof of Theorem 2. Let $S = \{a_\alpha : \alpha < \mathfrak{c}\}$ and let (n_k) be any increasing sequence of natural numbers. Then $S' := \{(a_\alpha(n_k))_{k \in \mathbb{N}} : \alpha < \mathfrak{c}\}$ is also a set of free generators in c_0. This easily follows from (3).

In [1] it was proved that the set E of all sequences from l^∞ which are not convergent is $\mathfrak{c}$-lineable, i.e. there is a linear $\mathfrak{c}$-dimensional subspace of l^∞ which every non-zero element is divergent. This part is devoted to some strengthening of this result to obtain a theorem on algebrability.

We say that a function $f : \mathbb{R} \to \mathbb{R}$ fulfils (N) Luzin condition if the images of null sets are null (i.e. of Lebesque measure zero). Note that all polynomials satisfy (N) Luzin condition and are finite-to-one.

Lemma 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a finite-to-one continuous function satisfying (N) Luzin condition. Then the image $f(C)$ of any compact perfect zero-dimensional set C is also compact perfect and zero-dimensional.*

Proof. Let $x \in f(C)$, $y \in f^{-1}(\{x\})$ and take $y_n \to y$ with $y_n \ne y$ and $y_n \in C$ for all n. Then the sequence $(f(y_n))$ tends to x, and it contains a subsequence all of whose elements are distinct from x. Therefore $f(C)$ is dense-in-itself. Clearly it is compact and thus $f(C)$ is perfect. Since $f(C)$ is null, it is also zero-dimensional. $\square$

For a sequence $x \in \mathbb{R}^\mathbb{N}$ let $\mathrm{LIM}(x)$ stands for the set of all limit points of x, i.e. $\mathrm{LIM}(x) = \{y \in \mathbb{R} : x(n_k) \to y$ for some $(n_k)\}$. Clearly if $x \in l^\infty$,

then $\mathrm{LIM}(x)$ is nonempty. Note that $x \in l^\infty$ is convergent if and only if $\mathrm{LIM}(x)$ is a singleton. In general, $\mathrm{LIM}(x)$ is a nonempty compact subset of $\mathbb{R}$. On the other hand if C is a non-empty compact subset of reals, then there is a decomposition of C into a perfect set $P \subset C$ (which can be empty) and a countable set $D \subset C$ (this is the famous Cantor–Bendixson theorem). Now, taking (x_n) such that each point of D appears in (x_n) infinitely many times, and the set of remaining points of (x_n) is dense in P, we obtain $\mathrm{LIM}(x) = C$. Let $E_{\mathrm{Cantor}} = \{x \in l^\infty : \mathrm{LIM}(x) \text{ is the union of a finite set}$ and a set homeomorphic to the Cantor set$\}$.

Let $A_1, A_2, \ldots$ be a partition of the set of even numbers in $\mathbb{N}$ into infinite sets. Let $B_i = A_i \cup \{2i - 1\}$. Then clearly $B_1, B_2, \ldots$ is a partition of $\mathbb{N}$. Let $x \in l^\infty$ be the characteristic function of even numbers. Then $\mathrm{LIM}(x) = \{0, 1\}$ and $\bigcup_{n=1}^\infty \mathrm{LIM}(x_{\upharpoonright B_n}) = \{1\}$. Therefore one cannot have the equality $\mathrm{LIM}(x) = \mathrm{cl}(\bigcup_{n=1}^\infty \mathrm{LIM}(x_{\upharpoonright B_n}))$ for any partition $B_1, B_2, \ldots$. However, we easily obtain the following.

Lemma 4. *Assume that* $\{x(n) : n \in \mathbb{N}\} \subset \mathrm{cl}(\bigcup_{n=1}^\infty \mathrm{LIM}(x_{\upharpoonright B_n}))$. *Then* $\mathrm{LIM}(x) = \mathrm{cl}(\bigcup_{n=1}^\infty \mathrm{LIM}(x_{\upharpoonright B_n}))$.

A family $\mathcal{F}$ of infinite subsets of $\mathbb{N}$ is called almost disjoint if $A \setminus B$ is finite for every $A, B \in \mathcal{F}$. It is well-known that there is an almost disjoint family of cardinality $\mathfrak{c}$. The classical Brouwer theorem states that a subset of $\mathbb{R}$ is homeomorphic to the Cantor set if and only if it is compact, dense-in-itself and zero-dimensional (cf. [15, Theorem 7.4]). Note also that a meager compact set is zero-dimensional.

Theorem 5. E_{Cantor} *is strongly* $\mathfrak{c}$*–algebrable.*

Proof. Let $\{a_\alpha : \alpha < \mathfrak{c}\}$ be a set of free generators of the algebra defined in the proof of Theorem 2. Let C be the Cantor set. Let $N_1, N_2, \ldots$ be a partition of $\mathbb{N}$ into infinitely many infinite sets. Let $\{F_\alpha^i : \alpha < \mathfrak{c}\}$ be an almost disjoint family of subsets of N_i. For $\alpha < \mathfrak{c}$ define $x_\alpha(n) = a_\alpha(i)$ if

$n \in N_i \setminus F_\alpha^i$, and define x_α on F_α^i such that $\mathrm{cl}\left(\{x_\alpha(n) : n \in F_\alpha^i\}\right) = a_\alpha(i) + C$. Now let A be the algebra generated by $\{x_\alpha : \alpha < \mathfrak{c}\}$.

Fix $x \in A \setminus \{0\}$. Then there exist $\alpha_1 < \alpha_2 < ... < \alpha_m < \mathfrak{c}$ as well as a non-trivial polynomial P of m variables with non-zero real coefficients such that $x = P(x_{\alpha_1}, x_{\alpha_2}, ..., x_{\alpha_m})$. Note that $P(a_{\alpha_1}(i), a_{\alpha_2}(i), ..., a_{\alpha_m}(i)) \neq 0$ for all but finitely many $i \in \mathbb{N}$, and

$$\lim_{i \to \infty} P(a_{\alpha_1}(i), a_{\alpha_2}(i), ..., a_{\alpha_m}(i)) = 0$$

Fix $i \in \mathbb{N}$ and $j \in \{1, ..., m\}$. Since $\{F_\alpha^i : \alpha < \mathfrak{c}\}$ is almost disjoint, there is $M \in \mathbb{N}$ such that $(F_{\alpha_j}^i \setminus \{1, ..., M\}) \cap \bigcup_{k \neq j} F_{\alpha_k}^i = \emptyset$. Hence for each $k \neq j$ we have $x_{\alpha_k}(n) = a_{\alpha_k}(i)$ for any $n \in F_{\alpha_j}^i \setminus \{1, ..., M\}$. Moreover $\mathrm{LIM}((x_{\alpha_j})_{\restriction F_{\alpha_j}^i \setminus \{1,...,M\}}) = a_{\alpha_j}(i) + C$. If $n \in N_i \setminus \bigcup_{k=1}^m F_{\alpha_k}^i$, then $x_{\alpha_k}(n) = a_{\alpha_k}(i)$. Consequently

$$\mathrm{LIM}(x_{\restriction F_{\alpha_j}^i \setminus \{1,...,M\}}) = \mathrm{LIM}(x_{\restriction F_{\alpha_j}^i}) = C_{ij},$$

where

$$C_{ij} = P(\{a_{\alpha_1}(i)\} \times ... \times \{a_{\alpha_{j-1}}(i)\} \times (a_{\alpha_j}(i) + C) \times \{a_{\alpha_{j+1}}(i)\} \times ... \times \{a_{\alpha_m}(i)\}).$$

Therefore by Lemma 4 we have

$$(4) \qquad \mathrm{LIM}(x) = \mathrm{cl}\left(\bigcup_{i=1}^\infty \bigcup_{j=1}^m C_{ij}\right).$$

Now, by Lemma 3 the set C_{ij} is homeomorphic to the Cantor set (and therefore of measure zero) if and only if the function f_{ij} define by the formula $f_{ij} = P(a_{\alpha_1}(i), ..., a_{\alpha_{j-1}}(i), t, a_{\alpha_{j+1}}(i), ..., a_{\alpha_m}(i))$ is not constant. If f_{ij} is constant, then $C_{ij} = \{P(a_{\alpha_1}(i), ..., a_{\alpha_m}(i))\}$ is a singleton.

Claim. *For all but finitely many i's there exists j such that C_{ij} is homeomorphic to the Cantor set.*

Proof of Claim. Suppose to the contrary, that there are infinitely many $i_1 < i_2 < ...$ such that for each j every function $f_{i_k j}$ is constant. By the observation we did after Theorem 2, $S' = \{(a_\alpha(i_k))_{k \in \mathbb{N}} : \alpha < \mathfrak{c}\}$ is a set of

free generators. Note that the derivative $f'_{i_k j}$ equals zero. Since each partial derivative P_j of polynomial P is also a polynomial and S' forms a set of free generators, then P_j equals zero. Therefore P is constant. But P does not have a constant term. Hence P equals zero, which is a contradiction. This proves the claim.

The closure of a null set is not necessary null. Hence we need to specify what exactly we add in the closure of the set on the right side of equation (4). Take any convergent sequence (x_n) from

$$\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{m} C_{ij}.$$

Let $x_0 = \lim_n x_n$. Without loss of generality we may assume that all elements of the sequence are in $\bigcup_{i=1}^{\infty} C_{i1}$. If there is n_0 such that all elements of the sequence are in $\bigcup_{i=1}^{n_0} C_{i1}$, so is x_0. Therefore we may assume that there is an increasing sequence of indices (k_n) with

$$x_n \in P((a_{\alpha_1}(k_n) + C) \times \{a_{\alpha_2}(k_n)\} \times ... \times \{a_{\alpha_m}(k_n)\}).$$

Thus there is a sequence (y_n) of elements of C such that

$$x_n = P(a_{\alpha_1}(k_n) + y_n, a_{\alpha_2}(k_n), ..., a_{\alpha_m}(k_n)).$$

Since C is compact, we may assume that y_n tends to some $y \in C$. Since $a_{\alpha_i} \in c_0$, then

$$x_0 = P(y, 0, ..., 0) \in P(C \times \{(0, ..., 0)\}).$$

Hence

$$\mathrm{cl} \left(\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{m} C_{ij} \right) = \bigcup_{i=0}^{\infty} \bigcup_{j=1}^{m} C_{ij},$$

where we put $a_{\alpha_i}(0) = 0$. Thus by (4) the set of limit points $\mathrm{LIM}(x)$ is a countable union of null sets. Since it is compact, it is also zero-dimensional. Note that for fixed i the intersection $C_{i1} \cap ... \cap C_{im}$ is non-empty, namely it contains $P(a_{\alpha_1}(i), ..., a_{\alpha_m}(i))$. Therefore by the Claim, all but finitely many sets of the form $\bigcup_{j=1}^{m} C_{ij}$ are homeomorphic to the Cantor set. Therefore

$\mathrm{LIM}(x)$ is the union of a finite set and a countable union of sets homeomorphic to the Cantor set. If we remove from $\mathrm{LIM}(x)$ all its isolated points (there are finitely many of them), then the remaining part is the union of dense-in-itself sets, and therefore, as a dense-in-itself, zero-dimensional and compact set, it is homeomorphic to the Cantor set. To sum up, $x \in E_{\mathrm{Cantor}}$, as required. $\qquad\square$

By $B(x, \varepsilon)$ we denote the closed ball with the radius $\varepsilon > 0$ centred at the point x. For $A \subset l^\infty$ put $\mathrm{LIM}(A) = \bigcup \{\mathrm{LIM}(x) : x \in A\}$.

Lemma 6. *Let $B_1 \supset B_2 \supset B_3 \supset ...$ be a sequence of closed balls in l^∞ such that the corresponding sequence of their radii tends to zero. Then $\mathrm{LIM}(\bigcap_n B_n) = \bigcap_n \mathrm{LIM}(B_n)$.*

Proof. The inclusion $\subset$ is trivial. As for the converse, suppose that $a \in \bigcap_n \mathrm{LIM}(B_n)$. There is $x_n \in B_n$ and $m_1^n < m_2^n < ...$ with $\lim_k x_n(m_k^n) = a$. Since the sequence of diameters of B_n's tends to zero, (x_n) is a Cauchy sequence in l^∞. Let $x = \lim_n x_n$. We will show that $a \in \mathrm{LIM}(x) \subset \mathrm{LIM}(\bigcap_n B_n)$.

Suppose to the contrary that there is $\varepsilon > 0$ and a positive integer m such that $|x(k) - a| \geq \varepsilon$ for any $k \geq m$. By hypothesis, there is n_0 such that $\mathrm{diam}(B_n) < \varepsilon/2$ for $n \geq n_0$. But $a \in \mathrm{LIM}(x_n)$ for any n; therefore for fixed $n \geq n_0$ we have $|x_n(k) - a| < \varepsilon/2$ for infinitely many k's. Hence for each fixed $n \geq n_0$ there are infinitely many k with

$$|x_n(k) - x(k)| \geq |x(k) - a| - |x_n(k) - a| \geq \varepsilon - \varepsilon/2 = \varepsilon/2.$$

But this contradicts the fact that x_n tends to x in l^∞. $\qquad\square$

The Banach-Mazur game is an infinite topological game. Fix a topological space X and a subset A of it. Two players play open sets: player I starts with some nonempty open subset U_1, player II responses with an open subset V_1 of U_1, then player I takes some open $U_2 \subset V_1$, and player II responses with an open subset V_2 of U_2, etc. The second player wins if $\emptyset \neq \bigcap_n V_n \subset A$.

The Banach-Mazur theorem states that player II has a winning strategy in this game if and only if A is co-meager in X, cf. [15, Theorem 8.33] and [16, Chap. 6].

Let $E_{\text{finite}} = \{x \in l^\infty : \{x(n) : n \in \mathbb{N}\} \text{ is finite}\}$. It is clear that E_{finite} is a dense $\mathfrak{c}$-generated algebra.

Theorem 7. *The set E_{Cantor} is co-meager in l^∞.*

Proof. To prove the co-meagerness of E_{Cantor} we will apply the Banach-Mazur game. We need to define a winning strategy for player II.

Player I plays an arbitrary nonempty open set $U_1 \subset l^\infty$. There is $x_1 \in l^\infty$ and $\varepsilon > 0$ with $B(x_1, \varepsilon) \subset U_1$. Observe that we may assume that $x_1 \in E_{\text{finite}}$ and $\text{LIM}(x_1)$ is not a singleton.

Let $\text{LIM}(x_1) = \{a_{1,1} < a_{1,2} < ... < a_{1,n_1}\}$. Let $\varepsilon_1 < \frac{1}{4}\min\{a_{1,i} - a_{1,i-1} : i = 2, ..., n_1\}$ and $0 < \varepsilon_1 < \varepsilon/2$. Then

$$B(x_1, \varepsilon_1) \subset B(x_1, \varepsilon) \subset U_1.$$

Note that $\text{LIM}(B(x_1, \varepsilon_1))$ equals to the disjoint union of intervals $[a_{1,i} - \varepsilon_1, a_{1,i} + \varepsilon_1]$. Player II plays $V_1 := B(x_1, \varepsilon_1) \subset U_1$. Additionally define $N_{1,i} = \{n : x_1(n) = a_{1,i}\}$.

Player I plays an arbitrary nonempty open set $U_2 \subset V_1$. There is $x_2 \in l^\infty$ and $\varepsilon > 0$ with $B(x_2, \varepsilon) \subset U_2$. We may assume that $x_2 \in E_{\text{finite}}$ and that each $\text{LIM}((x_2)_{\upharpoonright N_{1,i}})$ is not a singleton $(i = 1, ..., n_1)$.

Let $\text{LIM}(x_2) = \{a_{2,1} < a_{2,2} < ... < a_{2,n_2}\}$. Let $\varepsilon_2 < \frac{1}{4}\min\{a_{2,i} - a_{2,i-1} : i = 2, ..., n_2\}$ and $0 < \varepsilon_2 < \varepsilon_1/2$. Then

$$B(x_2, \varepsilon_2) \subset B(x_2, \varepsilon) \subset U_2.$$

Note that $\text{LIM}(B(x_2, \varepsilon_2))$ equals to the disjoint union of intervals $[a_{2,i} - \varepsilon_2, a_{2,i} + \varepsilon_2]$. Player II plays $V_2 := B(x_2, \varepsilon_2) \subset U_2$. Additionally define $N_{2,i} = \{n : x_2(n) = a_{2,i}\}$, etc.

Note that the closure of V_{n+1} is contained in V_n. Since l^∞ is complete and $\varepsilon_n \to 0$, $\bigcap_n V_n$ is nonempty and, in fact, it reduces to a point, say x_0.

Moreover in every step n of the game each subinterval of $\mathrm{LIM}(V_n)$ is split into at least two but finitely many subintervals of $\mathrm{LIM}(V_{n+1})$ and the set $\mathrm{LIM}(V_{n+1})$ is a union of finitely many compact intervals. Hence $\bigcap_n \mathrm{LIM}(V_n)$ is homeomorphic to the Cantor set. By Lemma 6 the set $\mathrm{LIM}(\bigcap_n V_n) = \mathrm{LIM}(x_0)$ is homeomorphic to the Cantor set, and therefore E_{Cantor} is a co-meager subset of l^∞. $\square$

The above results lead us to the following natural questions.

Problem 8.   1. *Is E_{Cantor} densely algebrable in l^∞?*

2. *Does $E_{\mathrm{Cantor}} \cup \{0\}$ contain a co-meager algebra?*

3. *Is the set $\{x \in l^\infty : \mathrm{LIM}(x) \text{ is homeomorphic to Cantor set}\}$, $\mathfrak{c}$-algebrable?*

At the end of this section we will show an example of spaceable set in l^∞, which is even not 1-algebrable. By $E \subset l^\infty$ denote the set of all sequences which do not attain their supremum. Gurariy and Quarta have shown in [14] that E is spaceable.

Theorem 9. *E is not 1-algebrable.*

Proof. Suppose to the contrary that $E \cup \{0\}$ contains a non-trivial algebra. Then there is $x \in E$ such that any linear combination of $x, x^2, x^3, \ldots$ is either in E or equals 0. Note that $x^2 \in E$. Let $a = \sup_n x^2(n) > 0$. Then $\sup_n a^{-1}x^2(n) = 1$ and $\inf_n a^{-1}x^2(n) \geq 0$. Consider the following polynomial of one variable

$$P(t) = 2t^3 - 3a^{-1}x^2(1)t^2.$$

It is easy to see that P attains its supremum on the unit interval $[0,1]$ at the point $a^{-1}x^2(1)$. Therefore the sequence $P(a^{-1}x^2)$ attains its supremum at $P(a^{-1}x^2)(1)$. $\square$

3. NON-MEASURABLE FUNCTIONS

The simplest way to get a non-measurable function is to take a characteristic function of a non-measurable set. The most common way to get a non-measurable set is to use the Vitali or the Bernstein constructions. A set $B \subset \mathbb{R}$ is called a Bernstein set provided that $B \cap P \neq \emptyset$ and $B^c \cap P \neq \emptyset$ for any perfect subset P of the real line.

Suppose that $\{B_\alpha < \mathfrak{c}\}$ is a decomposition of $\mathbb{R}$ into $\mathfrak{c}$ pairwise disjoint Bernstein sets. Let $S \subsetneq \mathfrak{c}$. Then $B := \bigcup_{\alpha \in S} B_\alpha$ is also a Bernstein set. To see this, fix $\alpha \in S$ and $\beta \in \mathfrak{c} \setminus S$. Then B_α and B_β have nonempty intersections with every perfect set P, and consequently their supersets B and B^c, respectively, also satisfy the same condition. Using this observation and [13, Theorem 2.2] we obtain that the set of all non-measurable functions in $\mathbb{R}^{\mathbb{R}}$ is $2^{\mathfrak{c}}$–lineable. Note that there exists a function $F : \mathbb{R} \to \mathbb{R}$ such that $F^{-1}(\{x\})$ is a Bernstein set for every $x \in \mathbb{R}$.

For any set X we can consider $X^{\mathbb{R}}$ as a linear algebra under pointwise addition and product. We say that a family $\{A_s : s \in S\}$ of subsets of X is independent if for any distinct $s_1, s_2, ..., s_n \in S$ and any $\varepsilon_i \in \{-1, 1\}$ for $i = 1, ..., n$ the set $A_{s_1}^{\varepsilon_1} \cap ... \cap A_{s_n}^{\varepsilon_n}$ is nonempty, where $B^1 = B$ and $B^{-1} = X \setminus B$ for any $B \subset X$. Note that for any set of cardinality κ there is an independent family of cardinality 2^κ (see [6]).

Lemma 10. *Let A be a commutative algebra with a unity. Let $x \in A$ be a free generator and let $P \neq 0$ be a polynomial of n variables without non-zero constant term. Then at least one of elements $P(x, 1, ..., 1)$, $P(1, x, 1, ..., 1)$,...,$P(1, ..., 1, x)$ is non-zero.*

Proof. Let P be a non-zero polynomial of the form

$$P(t_1, ..., t_n) = \beta_1 t_1^{k_{11}} \cdot t_2^{k_{12}} \cdots t_n^{k_{1n}} + ... + \beta_m t_1^{k_{m1}} \cdot t_2^{k_{m2}} \cdots t_n^{k_{mn}}$$

where $\beta_1, ..., \beta_m$ do not vanish simultaneously and the matrix $(k_{ij} : i \leq m, j \leq n)$ consists of non-negative integers with non-zero and distinct rows.

Suppose that each of $P(x, 1, ..., 1), P(1, x, 1, ..., 1), ..., P(1, ..., 1, x)$ is zero. Then

$$\beta_1 x^{k_{11}} + ... + \beta_m x^{k_{m1}} = 0.$$

Since x is a free generator, then $k_{11} = k_{21} = ... = k_{m1}$ and $\beta_1 + ... + \beta_m = 0$. Similarly we obtain $k_{1j} = k_{2j} = ... = k_{mj}$ for every j. Therefore each row of $(k_{ij} : i \leq m, j \leq n)$ equals to $(k_{11}, ..., k_{1n})$, a contradiction. □

Theorem 11. *Let X be a set of cardinality κ. Then the set of all functions from $\mathbb{R}^X$ which are not constant is strongly 2^κ-algebrable.*

Proof. We may assume that X is of the form $Y \times \mathbb{N}$ with Y of cardinality κ. Let $\{A_\alpha : \alpha < 2^\kappa\}$ be an independent family of subsets of Y. For any $\alpha < 2^\kappa$ define $f_\alpha : Y \times \mathbb{N} \to \mathbb{R}$ in the following way. If $x \notin A_\alpha$, put $f_\alpha(x, n) = 1$ for any $n \in \mathbb{N}$. If $x \in A_\alpha$, put $f_\alpha(x, n) = \frac{1}{\ln(n+1)}$. We will show that $\{f_\alpha : \alpha < 2^\kappa\}$ is a free system of generators of an algebra.

Take any $\alpha_1 < \alpha_2 < ... < \alpha_n < 2^\kappa$ and any nonzero polynomial P without free term. Consider a function $f := P(f_{\alpha_1}, ..., f_{\alpha_n})$. Note that f on $(A^1_{\alpha_1} \cap A^{-1}_{\alpha_2} \cap ... \cap A^{-1}_{\alpha_n}) \times \mathbb{N}$ is equal to $P(f_{\alpha_1}, 1, ..., 1)$. As it was shown in the proof of Theorem 2, the sequence $\left(\frac{1}{\ln(n+1)}\right)_{n=1}^\infty$ is a free generator of an algebra in c_0. Therefore by Lemma 10 we may assume that f is non-constant on the non-empty set $(A^1_{\alpha_1} \cap A^{-1}_{\alpha_2} \cap ... \cap A^{-1}_{\alpha_n}) \times \mathbb{N}$. The result follows. □

Finally, by $E \subset \mathbb{R}^\mathbb{R}$ we denote the set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f^{-1}(\{x\})$ is either an empty set or a Bernstein set for any $x \in \mathbb{R}$. Note that any function from E is Lebesgue non-measurable and has not the Baire property.

Theorem 12. *The set E is strongly $2^{\mathfrak{c}}$–algebrable.*

Proof. Let $F : \mathbb{R} \to \mathbb{R}$ be a function such that $F^{-1}(\{t\})$ is a Bernstein set for any $t \in \mathbb{R}$. Let $\{f_\alpha : \alpha < 2^{\mathfrak{c}}\}$ be a set of free generators in $\mathbb{R}^X$ given in Theorem 11. The family $\{f_\alpha \circ F : \alpha < 2^{\mathfrak{c}}\}$ generates a free algebra contained in $E \cup \{0\}$. □

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